SULLO SVILUPPO DI UN DETERMINANTE

NOTA

DEL PROF. FRANCESCO BRIOSCHI

Posto per brevità:

$$a_{r,s} = \frac{1}{x_s - a_r}, \quad \alpha_{r,s} = \frac{1}{(x_s - a_r)^2}.$$

si ha facilmente che il determinante:

$$A = \begin{Bmatrix} a_{1,1} & \alpha_{1,1} & \cdot & \cdot & \cdot & \cdot & a_{n,1} & \alpha_{n,1} \\ a_{1,2} & \alpha_{1,2} & \cdot & \cdot & \cdot & \cdot & a_{n,2} & \alpha_{n,2} \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ a_{1,2n} & \alpha_{1,2n} & \cdot & \cdot & \cdot & \cdot & a_{n,2n} & \alpha_{n,2n} \end{Bmatrix}$$

è uguale a:

$$(1) \qquad (-1)^n \frac{\Pi^4(a_1, a_2, \ldots a_n)\ \Pi(x_1, x_2, \ldots x_{2n})}{\varphi^2(a_1)\ \varphi^2(a_2) \ldots \varphi^2(a_n)}$$

essendo:

$$\varphi(x) = (x - x_1)\ (x - x_2) \ldots (x - x_{2n})$$

ed il simbolo $\Pi(a_1, a_2, \ldots a_n)$ indicando il prodotto delle differenze delle quantità $a_1, a_2, \ldots a_n$, cioè:

$$\Pi(a_1, a_2, \ldots a_n) = (a_n - a_1)\ (a_n - a_2) \ldots (a_n - a_{n-1})\ (a_{n-1} - a_1) \ldots (a_2 - a_1).$$

Ora supponendo:

$$P(x) = (x - a_1)\ (x - a_2) \ldots (x - a_n)$$

si hanno le relazioni:

$$\varphi(a_1)\ \varphi(a_2) \ldots \varphi(a_n) = P(x_1)\ P(x_2) \ldots P(x_{2n})$$

$$\Pi^2(a_1, a_2, \ldots a_n) = (-1)^{\frac{n(n-1)}{2}} P'(a_1)\ P'(a_2) \ldots P'(a_n)$$

dunque si avrà anche:

$$(2) \qquad A = (-1)^n . \frac{P'^2(a_1)\, P'^2(a_2) \ldots P'^2(a_n)\ \Pi(x_1, x_2, \ldots x_{2n})}{\varphi(a_1)\ \varphi(a_2) \ldots \varphi(a_n) \ldots P(x_1)\ P(x_2) \ldots P(x_{2n})}$$

Per ottenere gli sviluppi dei determinanti parziali $\frac{dA}{d\alpha_{r,s}}$, $\frac{dA}{da_{r,s}}$ osserviamo che:

$$(3)\quad \begin{cases} \dfrac{dA}{d\alpha_{r,s}} = (-1)^s \left\{ a_{r,1} A_{r,1} + \ldots + a_{r,s-1} A_{r,s-1} + a_{r,s+1} A_{r,s+1} + \ldots + a_{r,2n} A_{r,2n} \right\} \\ \dfrac{dA}{da_{r,s}} = (-1)^{s-1} \left\{ \alpha_{r,1} A_{r,1} + \ldots + \alpha_{r,s-1} A_{r,s-1} + \alpha_{r,s+1} A_{r,s+1} + \ldots \alpha_{r,2n} A_{r,2n} \right\} \end{cases}$$

essendo $A_{r,1}$, $A_{r,2}$ ... determinanti del $2(n-1)^{mo}$ ordine analoghi all'A. Quindi pel valore (1) di quest' ultimo determinante si avrà:

$$A_{r,i} = (-1)^{n+i} \frac{\Pi^4(a_1,..a_{r-1}, a_{r+1},..a_n)\,\Pi(x_1,..x_{i-1},x_{i+1},..x_{s-1}, x_{s+1},..x_{2n})}{f^2(a_1)\ldots f^2(a_{r-1})\, f^2(a_{r+1})\ldots f^2(a_n)} \qquad (i<s)$$

$$A_{r,i} = (-1)^{n+i-1} \frac{\Pi^4(a_1,..a_{r-1}, a_{r+1},..a_n)\,\Pi(x_1,..x_{i-1},x_{i+1},..x_{s-1}, x_{s+1},..x_{2n})}{f^2(a_1)\ldots f^2(a_{r-1})\, f^2(a_{r+1})\ldots f^2(a_n)} \qquad (i>s)$$

nelle quali:

$$f(x) = \frac{\psi(x)}{x-x_i}, \quad \psi(x) = \frac{\varphi(x)}{x-x_s}.$$

Ora ponendo:

$$N(x) = \frac{P(x)}{x-a_r}$$

si ha:

$$f^2(a_1)\ldots f^2(a_{r-1})\, f^2(a_{r+1})\ldots f^2(a_n) = \frac{\psi^2(a_1)\ldots\psi^2(a_{r-1})\,\psi^2(a_{r+1})\ldots\psi^2(a_n)}{N^2(x_i)}$$

inoltre:

$$\Pi(x_1,\ldots x_{i-1}, x_{i+1},\ldots x_{s-1}, x_{s+1},..x_{2n})\,\psi'(x_i) = (-1)^{i+1}\,\Pi(x_1, x_2, .. x_{s-1}, x_{s+1},..x_{2n}) \qquad (i<s)$$

$$\Pi(x_1,..x_{s-1}, x_{s+1},\ldots x_{i-1}, x_{i+1},..x_{2n})\,\psi'(x_i) = (-1)^{i}\,\Pi(x_1, x_2, .. x_{s-1}, x_{s+1},.. x_{2n}) \qquad (i>s)$$

quindi:

$$A_{r,i} = (-1)^{n-1} \frac{\Pi^4(a_1,\ldots a_{r-1}, a_{r+1},\ldots a_n)\,\Pi(x_1, x_2,\ldots x_{s-1}, x_{s+1},\ldots x_{2n})}{\psi^2(a_1)\ldots\psi^2(a_{r-1})\,\psi^2(a_{r+1})\ldots\psi^2(a_n)}\,\frac{N^2(x_i)}{\psi'(x_i)}.$$

Sostituendo questo valore nella prima delle (3), osservando essere:

$$\frac{N^2(x_1)}{(x_1-a_r)\,\psi'(x_1)}+\ldots+\frac{N^2(x_{s-1})}{(x_{s-1}-a_r)\,\psi'(x_{s-1})}+\frac{N^2(x_{s+1})}{(x_{s+1}-a_r)\,\psi'(x_{s+1})}$$

$$+\ldots+\frac{N^2(x_{2n})}{(x_{2n}-a_r)\psi'(x_{2n})}=-\frac{P'^2(a_r)}{\psi(a_r)}$$

si ottiene:

$$\frac{dA}{d\alpha_{r,s}}=(-1)^{n+s}\,\frac{\Pi^2(a_1,\ldots a_{r-1},a_{r+1},\ldots a_n)\,\Pi(x_1,x_2,\ldots x_{s-1},x_{s+1},\ldots x_{2n})}{\psi^2(a_1)\ldots\psi^2(a_{r-1})\,\psi^2(a_{r+1})\ldots\psi^2(a_n)}\,\frac{P'^2(a_r)}{\psi(a_r)}$$

ed analogamente:

$$\frac{dA}{da_{r,s}}=(-1)^{n+s-1}\,\frac{\Pi^2(a_1,\ldots a_{r-1},a_{r+1},\ldots a_n)\,\Pi(x_1,x_2,\ldots x_{s-1},x_{s+1},\ldots x_{2n})}{\psi^2(a_1)\ldots\psi^2(a_{r-1})\,\psi^2(a_{r+1})\ldots\psi^2(a_n)}\left(\frac{P'^2(a_r)}{\psi(a_r)}\right)'_{a_r}.$$

Queste espressioni dei determinanti parziali $\frac{dA}{d\alpha_{r,s}}$, $\frac{dA}{da_{r,s}}$ si ponno trasformare come si è fatto superiormente pel determinante A; infatti osservando che:

$$\Pi^2(a_1,\ldots a_{r-1},a_{r+1},\ldots a_n)=(-1)^{\frac{n(n-1)}{2}}.\frac{P'(a_1)\,P'(a_2)\ldots P'(a_n)}{P'^2(a_r)}$$

$$\psi(a_1)\,\psi(a_2)\ldots\psi(a_n)=(-1)^n\,P(x_1)\ldots P(x_{s-1})\,P(x_{s+1})\ldots P(x_{2n})$$

dalle superiori si dedurranno le seguenti:

$$(4)\quad\begin{cases}\dfrac{dA}{d\alpha_{r,s}}=(-1)^s\,\dfrac{P'^2(a_1)P'^2(a_2)\ldots P'^2(a_n)}{\psi(a_1)\psi(a_2)\ldots\psi(a_n)}.\,\dfrac{\Pi(x_1,\ldots x_{s-1},x_{s+1},\ldots x_{2n})}{P(x_1)\ldots P(x_{s-1})\,P(x_{s+1})\ldots P(x_{2n})}.\,\dfrac{\psi(a_r)}{P'^2(a_r)}\\[2ex] \dfrac{dA}{da_{r,s}}=(-1)^s\,\dfrac{P'^2(a_1)P'^2(a_2)\ldots P'^2(a_n)}{\psi(a_1)\,\psi(a_2)\ldots\psi(a_n)}.\,\dfrac{\Pi(x_1,\ldots x_{s-1},x_{s+1},\ldots x_{2n})}{P(x_1)\ldots P(x_{s-1})P(x_{s+1})\ldots P(x_{2n})}\left(\dfrac{\psi(a_r)}{P'^2(a_r)}\right)'_{a_r}\end{cases}$$

Pavia Ottobre 1857.